A Compositional Model of Fuzzy Concepts in Conceptual Spaces

Sean Tull[1 [0000−0001−5131−1464]]

Cambridge Quantum Computing, 17 Beaumont Street, OX1 2NA, Oxford
sean.tull@cambridgequantum.com

Keywords: Conceptual Space, Fuzzy Concept, Log-Concave, Category

How can we model conceptual reasoning in a way which is formal and yet reflects the fluidity of concept use in human cognition? One answer to this question is given by Peter Gärdenfors' framework of *conceptual spaces* [5, 6], in which domains of conceptual reasoning are modelled by mathematical spaces and concepts are described geometrically, typically as *convex* regions. Many authors have given their own formalisations of this framework, e.g. [1, 9, 2].

A notable aspect of the conceptual space framework is that it is *compositional* in the sense that each overall conceptual space is given by composing various simpler *domains* (e.g. colour, sound, taste). A highly successful approach to formalising such compositional theories of semantics is through the use of *category theory*, and particularly *monoidal categories*, as are used for example in the *DisCoCat* framework for NLP [4].

Bolt et al. [3] have presented such a categorical formalisation of conceptual spaces using the category of *convex relations*, with conceptual spaces modelled as *convex algebras* and concepts as convex subsets. The model demonstrates the naturality of monoidal categories for modelling the composition of domains, and for describing correlations between domains within concepts.

However, like most formalisations, the model of [3] is limited to describing only what we may call *crisp* concepts, which are such that any entity either strictly is or is not a member. In contrast, the cognitive science literature acknowledges that concepts should be *fuzzy* in that for each concept C the degree of membership of an entity x should form a scalar value $C(x) \in [0, 1]$. For example, Gärdenfors suggests defining membership based on distance from a central *prototypical* region [6]. A recent formalisation of conceptual spaces which does treat fuzziness from the outset is that of Bechberger and Kühnberger [2].

In this work we propose a compositionally well-behaved formalisation of fuzzy concepts, as *log-concave* functions $C \colon X \to [0, 1]$. We prove that these are essentially the smallest class of functions which are closed compositionally and which satisfy the criterion of *quasi-concavity*, identified implicitly by Gärdenfors.

Furthermore we specify a symmetric monoidal category **LCon** whose objects are conceptual spaces and morphisms are suitable probabilistic *conceptual mappings* between them, with fuzzy concepts as a special case. Thus we extend the model of Bolt et al. [3] from crisp to fuzzy concepts. This situates our category as a model of fuzzy conceptual spaces within the DisCoCat framework for NLP [4].

Fuzzy Concepts

In this work, we follow a simple model of a conceptual space as a *convex space*, namely a set X in which we may take convex combinations $\sum_{i=1}^n p_i x_i$ of elements, and which comes with a σ*-algebra* Σ_X, allowing us apply the tools of probability theory. A *crisp concept* of X is then a (measurable) subset $A \subseteq X$ which is *convex*, meaning it is closed under convex combinations.

More general fuzzy concepts should form certain maps $C \colon X \to [0,1]$. Gärdenfors implicitly proposes in [6] that fuzzy concepts should satisfy *quasi-concavity*:

$$C(p \cdot x + (1-p)y) \geq \min\{C(x), C(y)\}$$

This means that any point 'between' x and y is at least as much an instance of the concept C as they both are. However, such functions are in general not closed compositionally, i.e. under products, and so we will require further criteria.

We define a *fuzzy concept* on X to be a measurable map $C \colon X \to [0,1]$ which is *log-concave*, meaning that for all $x, y \in X$ and $p \in [0,1]$ we have

$$C(x +_p y) \geq C(x)^p C(y)^{1-p}$$

Log-concave functions are well-studied in economics, statistics and measure theory, and include many common statistical functions [10]. Crucially they contain every crisp concept A via its indicator function $C = 1_A$, every function which is affine, meaning that $C(\sum p_i x_i) = \sum_i p_i C(x_i)$, and every multi-variate Gaussian function. In a metric space, the latter allows us to naturally model fuzzy concepts with value inversely proportional to distance from some given prototypical convex region A, given by $C(x) = \lambda^{d(x,A)^n}$ for $\lambda \geq 1$, $n \in \mathbb{N}$.

Our definition is justified by the following result. It tells us that if we wish fuzzy concepts to be quasi-concave, compositionally well-behaved, and contain a few basic examples, then log-concavity is necessary.

Theorem 1. *Log-concave functions are the broadest class of quasi-concave functions $X \to [0,1]$ on each space X, which together are closed under products and contain all affine maps (and/or exponential maps).*

Fuzzy Conceptual Processes

We now wish to extend our definition to formalisation a notion of fuzzy *conceptual mapping* $f \colon X \to Y$ between conceptual spaces. Examples of mappings include reasoning processes on spaces, and metaphors or analogies, which may be viewed as mappings between distinct domains.

The fuzzy setting suggests we begin from the standard notion of a probabilistic map, namely a *Markov kernel* or *channel* $f \colon X \to Y$, which sends each $x \in X$ to a (sub)-probability measure $f(x)$ over Y, the general mathematical definition of a probability distribution. We say that a channel $f \colon X \to Y$ is *log-concave*, or a *conceptual channel*, when it satisfies

$$f(px + (1-p)y, pA + (1-p)B) \geq f(x, A)^p f(y, B)^{1-p} \tag{1}$$

for all $x, y \in X$, $p \in [0,1]$ and convex measurable $A, B \subseteq Y$.

Let I be the trivial space. As special cases, a conceptual channel $I \to X$ is essentially a *log-concave measure* on X [8], and models a probabilistic or 'noisy' state of X. These include many standard distributions such as Gaussians, point and uniform distributions. A conceptual channel $X \to I$ (called an *effect*) is a fuzzy concept on X, and more general channels are transformations of these.

Our main technical result is that such channels are compositionally well-behaved, forming a *symmetric monoidal category* **LCon**. This allows us to use them as a model of natural language in the DisCoCat framework, giving a fuzzy conceptual meaning to a sentence whose individual words are given meanings as fuzzy concepts. [4]. Moreover, this category is in a sense canonical, assuming a mild measure-theoretic condition on our convex spaces.

Theorem 2. *Log-concave channels form a symmetric monoidal category* **LCon**. *Moreover any other symmetric monoidal category* **C** *of 'well-behaved' spaces and channels whose effects are quasi-concave and contains crisp concepts and affine functions has an embedding* $\mathbf{C} \hookrightarrow \mathbf{LCon}$.

Future Work Our work has established a theoretical foundation for the compositional study of fuzzy concepts on convex spaces. In future we hope to explore the usefulness of this as model for probabilistic and fuzzy conceptual reasoning, in both NLP and AI. In particular, the fact that the category **LCon** allows us to model reasoning with noisy (probabilistic) inputs should make it applicable to modern neural net systems which use such noisy inputs, such as β-VAEs [7].

References

1. Aisbett, J., Gibbon, G.: A general formulation of conceptual spaces as a meso level representation. Artificial Intelligence **133**(1-2), 189–232 (2001)
2. Bechberger, L., Kühnberger, K.U.: A thorough formalization of conceptual spaces. In: Joint German/Austrian Conference on Artificial Intelligence (Künstliche Intelligenz). pp. 58–71. Springer (2017)
3. Bolt, J., Coecke, B., Genovese, F., Lewis, M., Marsden, D., Piedeleu, R.: Interacting conceptual spaces i: Grammatical composition of concepts. In: Conceptual Spaces: Elaborations and Applications, pp. 151–181. Springer (2019)
4. Coecke, B., Sadrzadeh, M., Clark, S.: Mathematical foundations for a compositional distributional model of meaning. arXiv preprint arXiv:1003.4394 (2010)
5. Gärdenfors, P.: Conceptual spaces: The geometry of thought. MIT press (2004)
6. Gärdenfors, P.: The geometry of meaning: Semantics based on conceptual spaces. MIT press (2014)
7. Higgins, I., Matthey, L., Pal, A., Burgess, C., Glorot, X., Botvinick, M., Mohamed, S., Lerchner, A.: beta-vae: Learning basic visual concepts with a constrained variational framework (2016)
8. Klartag, B., Milman, V.: Geometry of log-concave functions and measures. Geometriae Dedicata **112**(1), 169–182 (2005)
9. Rickard, J.T., Aisbett, J., Gibbon, G.: Reformulation of the theory of conceptual spaces. Information Sciences **177**(21), 4539–4565 (2007)
10. Saumard, A., Wellner, J.A.: Log-concavity and strong log-concavity: a review. Statistics surveys **8**, 45 (2014)