A Concept Inventory for Teaching Introductory Mathematics

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The background of this paper is educational research. I am teaching an introductory mathematics class to first-year computer science students which employs a variety of contemporary teaching methods (interactive engagement, flipped-classroom, automatically evaluated interactive exercises and so on). Nevertheless 15-30% of the students still fail the class. According to [2], students find the transition from school- to expert-level competence of mathematics difficult because it encompasses a shift from mastering standardised procedures to employing mathematical concepts and mathematical thinking. Or, in other words, a shift from calculating to modelling and comprehension.

In other disciplines, concept inventories (CIs) present a pedagogical instrument for measuring conceptual learning of core concepts in form of standardised questionnaires[1]. Final exams assess some mathematical competencies, such as calculating or programming. But expert competencies, such as developing proofs or modelling complex applications may be too challenging for first-year students and are more difficult to assess in exams. Physics education research has shown that students may be able to pass exams by applying procedural problem solving strategies without actually understanding the core concepts of the subject matter [6]. For example, combinatorial problems can be solved by determining which type a problem belongs to and then applying a formula without any deeper understanding. Such procedural skills are undoubtedly part of mathematical expertise but not the only component. Students also need to acquire conceptual understanding and further skills. CIs provide a means for measuring conceptual development by comparing the students' answers on a pretest before the semester with their answers on a posttest at the end. But in mathematics, CIs are not readily available [6].

CIs for introductory physics are successful partly because many relevant concepts can be expressed in natural language (e.g. force, speed, temperature and distance). A CI then detects misconceptions where everyday assumptions are incorrectly applied to scientific concepts. Attempts have been made to also provide CIs for computing and mathematics but CIs appear to be more elusive in those domains [6]. A suitable list of CI concepts is more difficult to obtain because of the rapidly changing nature of computing topics, more disagreement about which topics to select and more reliance on terminology and formalisms which cannot be expressed in natural language and thus be evaluated independently of the students' prior educational backgrounds [6]. This paper argues that the difficulties of creating a CI for mathematics may be overcome if instead of focusing on mathematical content, the emphasis is placed on reasoning and

[1] A concept in a CI is expressed through multiple choice questions where a true statement about a concept is contrasted with distractors expressing common misconceptions.

the nature and purpose of mathematics. Moore [4] considers perceptions of the nature of proofs and expertise in logic and proof methods as some of the main problem areas for first-year university students (alongside problem-solving skills, learning mathematical language and understanding concepts). These skills pertain to all mathematical topics and, in particular, reasoning ability presents a skill that is (or should) also be present in everyday activities.

Therefore, this paper proposes two foundational concepts for a CI for introductory mathematics: (C1) an understanding of the nature of mathematical concepts and (C2) an understanding of how to establish truth (reflecting the role of proofs). Because C1 and C2 do not represent mathematical content but belong to a metalevel, it may be easier to obtain a consensus about them amongst mathematics teachers. Evidence for misconceptions that students have about C1 and C2 is provided by their behaviour: not seeing definitions and theorems as the main reference points for mathematical problems, expressing inappropriate intuitive explanations for mathematical concepts and not appreciating the value of proofs and correct logical reasoning (cf. [1] and [4]). According to Priss [5], strong contributing factors for C1 are: not perceiving mathematical concepts as formal (i.e. with precise extension and intension), abstract (i.e. entirely embedded within mathematics without reference to an external reality) and semiotically anonymous (i.e. independent of the exact vocabulary and formulas used because many equivalent definitions may exist for a concept). Consequently, subconcepts of C1 can be formulated as: C1.1 the role of definitions, C1.2 identifying properties of concepts (especially rule-based properties that cannot be intuitively imagined), C1.3 identifying extensions of concepts (and not confounding concepts with their prototypical examples), C1.4 understanding equivalence (for example of definitions).

Subconcepts of C2 pertain to the role and nature of proofs. A variety of different types of proofs is usually explicitly taught to students who major in mathematics. But at least at some tertiary institutions, mathematics for computing students is not taught using the traditional sequence of definitions, theorems and proofs. In that case only "easy" proofs are included, more difficult ones are omitted or replaced by explanations. It would be an interesting question to compare the impact of a teaching style with or without proofs. But Moore's [4] observations appear to indicate that even including proofs in the teaching material does not guarantee that students develop a desirable perception of proofs and proof skills. Nevertheless, independently of how mathematics is taught, there is an expectation that students should acquire reasoning skills. Subconcepts of C2 could be formulated corresponding to different types of proofs. But focusing on the metalevel, the following subconcepts are proposed in this paper: C2.1 logical soundness of proofs (excluding, for example, incomplete induction or abduction), C2.2 proofs as solely logical (and not, for example, causal) and independent of psychological and emotional factors (because intuition, comprehension and belief are neither necessary nor sufficient conditions for determining truth).

The concept C2 also raises philosophical questions. In primary and secondary education, mathematics is usually taught by encouraging a development of mental concept images which highlight connections between mathematical formalisms and everyday experience. But in tertiary education, mathematics is presented as a formal system that is entirely founded in logic. Contemporary educational theories tend to be based on a constructivist philosophy which emphasises the consensual mental construction of mathematical concepts as explanatory models of sensory experience. A Peircean pragmatist philosophy, however, which agrees with consensual mental construction insists also that truth can only be established by applying a scientific method of inquiry to concepts that are grounded in reality (even though in the domain of mathematics the reality solely consists of definitions). Therefore, from a pragmatist viewpoint two further subconcepts of C2 can be formulated as C2.3 the fallibility of truth and C2.4 the establishment of truth by a scientific community of inquiry. A single person writing a proof may not be sufficient for establishing truth because a scientific community still needs to re-evaluate and check any proofs developed by individuals. A community that does not adhere to scientific inquiry, for example a social networking community, is equally insufficient. While some mathematicians might oppose a pragmatist approach, a simple example of a community of inquiry consists of students that are reporting errors that occasionally occur even in textbooks and other class materials.

A CI for mathematics could then continue with non-metalevel concepts. According to Lakoff & Johnson [3], mathematical concepts often model some everyday experience, such as counting, measuring, two-dimensional representations, ordering, arranging, structuring and so on. Many of these can be expressed in natural language. Mainly relevant to introductory mathematics are questions about the precision and absoluteness of mathematical structures, the purpose of modelling, questions relating to emptiness, equivalence, equality, infinity, limits, combinatorial counting, logical reasoning involving implication, counter examples and quantifiers and, last but not least, a list of different types of proofs.

I suspect, however, that C1 and C2 and their subconcepts may be more important than any of the other possible concepts because as metalevel concepts they are relevant for all mathematical activity. Therefore I am currently in the process of developing a CI for C1 and C2 which I hope to test and evaluate during the next semester. I would present more details about the development at the workshop.

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